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On Temperature Field Evolution in Planetary Atmospheric Processes

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The nonlinear equation of the evolution of temperature field perturbations in global-scale processes is derived for the vertically averaged baroclinic atmospheric model. In the adiabatic approximation this equation reduces to the equation for a simple wave in the mean temperature of vertical air columns. The influence of nonadiabatic factors is also taken into account.

1. In a paper by one of the authors [1] the following equations were derived:

$$\frac{\partial m}{\partial t}-\frac{\beta R}{f_0^2}\frac{\partial (Tm)}{\partial x}=0,$$

$$\frac{\partial T}{\partial t}-\gamma\frac{\beta R}{f_0^2}\frac{T}{m}\frac{\partial (Tm)}{\partial x}+\frac{(1+\gamma)R}{f_0 m}(mT, T)=Q+\mu\nabla^2 T, \qquad (1)$$

which are similar in form to the "simplified system" of equations suggested by I.A. Kibel' [2] for the purpose of describing the dynamics of ultralong waves (i.e., waves with zonal wave numbers from 1 to 4) in the earth's atmosphere. Here $m = p/p_0$, $p_0 \approx 1000$ mbar, p is the surface pressure, f_0 is the average value of the Coriolis parameter, $\beta = (df/dy)_0$ is the average value of the meridional gradient of the Coriolis parameter, T is the weighted average temperature of an air column, R is the specific gas constant for air, c_p is the specific heat of air at constant pressure, $\gamma = R/c_p$, $(A, B) = (\partial A/\partial x)\partial B/\partial y - (\partial A/\partial y)\partial B/\partial x$, x and y are Cartesian coordinates whose axes are directed toward the east and north, respectively. These equations are obtained by averaging the initial hydrothermodynamic equations over altitude, similarly to what was done in [3, 4], and replacing the equations of motion by the geostrophic relations (in doing this it was taken into account that the Kibel' number for planetary-scale motions is equal to $O(10^{-2})$, i.e., it is an order of magnitude smaller than for the motions associated with synoptic processes). In addition, the nonadiabatic heat influxes (Q) and also (in order that the system be balanced energetically) the horizontal large-scale heat diffusion with the corresponding heat conductivity coefficient μ are taken into account in the model.

The study of the linearized equations of the system (1) that was made in [1] made it possible to formulate two basic conclusions: 1) the system (1) describes two different types of waves: a) fast waves, traveling westward with velocities of the order of 100 m/sec and representing the limiting case of Rossby waves with the large-scale compressibility of the atmosphere taken into account, b) relatively slow waves, moving eastward with velocities of the order of 10 m/sec and caused in a first approximation by the transport of entropy perturbations by the mean zonal flow (see also [4]; 2) due to the action of a given geographic distribution of nonzonal heat sources and sinks the system tries to adapt to the state, in which the longitudinal variations of the fields of the surface pressure m and the mean air column temperature T are in opposite phase. In this case, near such a balanced state the amplitude of the fast-wave component will be at least an order of magnitude less than for the slow.

Taking into account the foregoing, as well as the fact that in the real atmosphere the variation of ambient conditions occurs quite smoothly, we can conclude that the fast component of the ultralong waves exerts no noticeable influence on the long-period (of the order of 2-3 weeks and more) evolution of the system, and therefore it is desirable to filter out this component from the very outset, thereby simplifying the initial equations (1).

2. Let us write the system (1) in dimensionless form, choosing as the characteristic length $L = 10^7$ m, the characteristic time L/U (U = 15 m/sec is the mean zonal wind velocity), the characteristic value of $m - m_{avg} = 1$ and, finally, the characteristic value of $T - T_{avg} = 250$ K. We make the deviatives of T and m with respect to x and y dimensionless in accordance with the relations for the zonal and meridional components of the wind velocity $u = -\frac{R}{f_0 m}\frac{\partial (Tm)}{\partial y}$ and $v = \frac{R}{f_0 m}\frac{\partial (Tm)}{\partial x}$; we make derivatives with respect to t dimensionless as derivatives with respect to x, multiplied by the quantity U. As a result we obtain the system

$$\varepsilon\frac{\partial \tilde{m}}{\partial \tilde{t}}-\frac{\partial (\tilde{m}\tilde{T})}{\partial \tilde{x}}=0, \qquad (2)$$

$$\varepsilon\frac{\partial \tilde{T}}{\partial \tilde{t}}-\gamma\frac{\tilde{T}}{\tilde{m}}\frac{\partial (\tilde{m}\tilde{T})}{\partial \tilde{x}}+\frac{\varepsilon(1+\gamma)}{\tilde{m}}(\tilde{m}\tilde{T},\tilde{T})=\varepsilon\left[\tilde{\mu}\left(\frac{\partial^2 \tilde{T}}{\partial \tilde{x}^2}+\frac{\partial^2 \tilde{T}}{\partial \tilde{y}^2}\right)+\tilde{Q}\right], \qquad (3)$$

where all the quantities that have been made dimensionless are denoted by a tilde, $\varepsilon = Uf_0^2/\beta RT_{avg} = O(10^{-1})$ is a small parameter, equal to the ratio of the velocities of the slow and fast wave component, $\tilde{\mu} = \mu\beta/Uf_0$, $\tilde{Q} = QR/f_0U^2$. Strictly speaking, we cannot determine precisely *a priori* the order of magnitude of the terms, inside the brackets on the right side of (3), since we know neither the exact value of Q and μ nor the behavior of the second derivatives. Let us assume, however, that these terms are of the order of O(1); this assumption, first of all, agrees with the general character of the behavior of the actual fields and, secondly, as we shall see later, it is justified by the results obtained.

We write (2) and (3) as:

$$\tilde{m}=\tilde{m}_0+\varepsilon\tilde{m}_1+\varepsilon^2\tilde{m}_2+\ldots,\quad \tilde{T}=\tilde{T}_0+\varepsilon\tilde{T}_1+\varepsilon^2\tilde{T}_2+\ldots \tag{4}$$

Let us substitute (4) into (2) and (3) and equate the coefficients for the zero and first powers of ε:

$$\frac{\partial(\tilde{m}_0\tilde{T}_0)}{\partial\tilde{x}}=0, \tag{5}$$

$$\frac{\partial\tilde{m}_0}{\partial\tilde{t}}-\frac{\partial(\tilde{m}_0\tilde{T}_1+\tilde{m}_1\tilde{T}_0)}{\partial\tilde{x}}=0, \tag{6}$$

$$\frac{\partial\tilde{T}_0}{\partial\tilde{t}}-\gamma\frac{\tilde{T}_0}{\tilde{m}_0}\frac{\partial(\tilde{m}_0\tilde{T}_1+\tilde{m}_1\tilde{T}_0)}{\partial\tilde{x}}+\frac{(1+\gamma)}{\tilde{m}_0}(\tilde{m}_0\tilde{T}_0,\tilde{T}_0)_{-}= =\tilde{Q}+\tilde{\mu}\left(\frac{\partial^2\tilde{T}_0}{\partial\tilde{x}^2}+\frac{\partial^2\tilde{T}_0}{\partial\tilde{y}^2}\right). \tag{7}$$

Equation (5) is integrated to:

$$\tilde{m}_0\tilde{T}_0=\tilde{\varphi}(\tilde{y},\tilde{t}), \tag{8}$$

with $\varphi(\tilde{y}, \tilde{t})$ being an undefined function for now. Eliminating $\tilde{m}_0T_1 + \tilde{m}_1T_0$ from (6) and (7) and replacing $\tilde{m}_0$ by its expression in terms of φ and T_0 in accordance with (8), we obtain (returning to dimensioned variables and omitting the index on T):

$$\frac{\partial T}{\partial t}-\frac{R}{f_0}\frac{\partial\ln\varphi}{\partial y}T\frac{\partial T}{\partial x}-\frac{\gamma}{1+\gamma}T\frac{\partial\ln\varphi}{\partial t} =\frac{1}{1+\gamma}(Q+\mu\nabla^2T). \tag{9}$$

Let us show that the function φ is uniquely defined in terms of the temperature T. To do this let us note that $\varphi = \overline{\varphi} = \overline{mT}$, where the line above the symbols denotes an averaging over longitude. We assume that $\overline{m} = 1$, i.e., the average surface wind is zero. (This is the reason we excluded surface friction of the atmosphere on the earth and we assume that the system is balanced automatically in terms of angular momentum.) With (8) taken into account we rewrite this relation in the form

$$\varphi=\overline{[\overline{T}+(T-\overline{T})][1+(\varphi/T-1)]},$$

from which it is easy to obtain

$$\varphi=(\overline{T^{-1}})^{-1}=\overline{T}\Big/\left(1+\frac{\overline{T'^2}}{\overline{T}^2}+O\left(\frac{\overline{T'^3}}{\overline{T}^3}\right)\right), \tag{10}$$

where $T' = T - \overline{T}$.

Equation (9) together with the integral relation (10) forms the integrodifferential equation, first order in time, that we are seeking, which describes the evolution of the temperature field in the slow component of the ultralong waves. Below, it is convenient to convert from this equation to equations for $\overline{T}$ and T'. By averaging (9) over x and taking into account that T is a periodic function in terms of the longitude with a period L_0 (L_0 is the length of the circle of latitude perimeter), we obtain

$$\frac{\partial\overline{T}}{\partial t}-\frac{\gamma}{1+\gamma}\overline{T}\frac{\partial\ln\varphi}{\partial t}=\frac{1}{1+\gamma}\left[\overline{Q}+\mu\frac{\partial^2\overline{T}}{\partial y^2}\right]. \tag{11}$$

Then for T' we will have

$$\frac{\partial T'}{\partial t}-\frac{R}{f_0}\frac{\partial\ln\varphi}{\partial y}\overline{T}\frac{\partial T'}{\partial x}-\frac{R}{f_0}\frac{\partial\ln\varphi}{\partial y}T'\frac{\partial T'}{\partial x}- -\frac{\gamma}{1+\gamma}T'\frac{\partial\ln\varphi}{\partial t}=\frac{1}{1+\gamma}[Q-\overline{Q}+\mu\nabla^2T'], \tag{12}$$

multiplying this by T', T'^2, etc., and averaging over x, we obtain equations for $\overline{T'^2}$, $\overline{T'^3}$, etc. These equations, together with Eq. (11) and the relation (10), form in principle a closed system of an infinite number of differential equations.

3. Let us examine our problem first in the adiabatic approximation, i.e., we set the right sides in Eqs. (11) and (12) equal to zero. It is easy to show that in this case $\partial(\overline{T'^n}/\overline{T}^n)/\partial t = 0$ ($n \geq 2$). (To do this it is sufficient to multiply Eq. (12) by $\overline{T}\,T'^{n-1}$, average over x and then subtract from the resulting equation Eq. (11), multiplied by $\overline{T'^n}$.)

Taking this into account, as well as the fact that x has the form (10), we obtain from Eq. (11) $\partial\overline{T}/\partial t = 0$. Hence $\partial\overline{T'^n}/\partial t = 0$ ($n \geq 2$) and $\partial\varphi/\partial t = 0$ and, consequently, Eq. (12) in the adiabatic approximation is reduced to a very simple nonlinear wave equation:

$$\frac{\partial T'}{\partial t}+c(T')\frac{\partial T'}{\partial x}=0, \tag{13}$$

where

$$c(T')=-\frac{R}{f_0}\overline{T}\frac{\partial \ln \varphi}{\partial y}(1+T'/\overline{T}).$$

The dependence of c on T' leads to a typical nonlinear effect—a twisting of the profile of the propagating wave and, as a consequence, to the formation of discontinuities. Let us assume, for example, the original T' profile is the sinusoid $T'(x)|_{t=0} = A \sin(2\pi x/\lambda)$ with an amplitude A = 5 K and a wavelength $\lambda = 1.5 \times 10^7$ m (the chosen λ value corresponds to a wave with a zonal wave number $\ell = 2$ at 48° latitude) and let us assume, moreover, the average zonal wind velocity is $U = -\frac{R}{f_0}\overline{T}\partial \ln \varphi/\partial y = 15$ m/sec. Then $c|_{t=0} = U + u \sin(2\pi x/\lambda)$, u = 0.3 m/sec. It can be shown (see, for example, [5, chap. 2, § 8]) that for the chosen parameter values a discontinuity begins with zero magnitude at the instant of time $t_1 = \lambda/2\pi\mu \approx 93$ days and reaches a maximum value of 10 K at the time $t_2 = \lambda/4\mu \approx 146$ days. At this time the sinusoidal wave is transformed into a sawtooth shape, whose amplitude for $t \gg \lambda/\mu$ then decays proportionally to λt^{-1}. Let us note that the decay of the sawtooth wave is not in fact contradictory to the earlier conclusion that the value of $\overline{T'^n}$ ($n \geq 2$) must be conserved since this conservation law is satisfied only until the moment of discontinuity formation. The appearance of the discontinuity, as is known (see [6]), however, is accompanied by the appearance of a heat flow through it, leading to an equalization of the temperature perturbations on both sides of the discontinuity. In the case of a more exact—with dissipation taken into account—statement of the problem this corresponds to a marked increase in the absorption of wave energy in the region where twisting of the wavefront occurs.

4. Let us turn to an examination of Eqs. (11) and (12) with a nonzero right side. We shall consider heat influxes (both zonal and nonzonal), taken in Newtonian form: $\overline{Q} = (H(T^* - \overline{T})$ and $Q' = Q - \overline{Q} = S)T^{**} - T') - HT'$, where the coefficients H and S have the dimensions of reciprocal time; it is assumed that H = S = 8×10^{-7} sec$^{-1} \approx$ (16 days)$^{-1}$[7], $T^* = T^*(y,t)$ is some known function, the explicit form of which we do not need; $T^{**} = T^{**}(x,y,t) = \tau(y,t)\sin(4\pi x/L_0)$, where the quantity τ is chosen from the considerations that in a real atmosphere $|S(T^{**} - T')| \lesssim |H(T^* - \overline{T})|$ and therefore we assume τ = 10 K.

On the right side of Eq. (10) we ignore all terms, starting with $T'^2/\overline{T}^2$ since in the substitution of φ into (11) and (12) the discarded terms would be at least two orders of magnitude smaller than the rest of the terms in the equations. In addition, in (12) we ignore the term $\mu\partial^2 T'/\partial y^2$ compared to $\mu\partial^2 T'/\partial x^2$ since the basic dependence of the temperature field on latitude has already been taken into account in Eq. (11).

With the foregoing taken into account, Eqs. (11) and (12) become

$$\frac{\partial \overline{T}}{\partial t} = H(T^* - \overline{T}) + \mu\frac{\partial^2 \overline{T}}{\partial y^2}, \tag{14}$$

$$\frac{\partial T'}{\partial t} - \frac{R}{f_0}\frac{\partial \overline{T}}{\partial y}\left(1+\frac{T'}{\overline{T}}\right)\frac{\partial T'}{\partial x} = S(T^{**} - T') - HT' + \mu\frac{\partial^2 T'}{\partial x^2}. \tag{15}$$

Solving Eq. (14), we can find the function $\overline{T}(y,t)$, which in turn enters parametrically into the nonlinear term in Eq. (15). A study of Eq. (15) in general form is difficult and therefore we shall examine two special cases in the following paragraphs.

5. Let us assume S = 0, i.e., there are no nonzonal heat influxes. We take the initial temperature profile in the form $T(x)|_{t=0} = A\cos(2\pi x/\lambda)$. The obvious corollary of (15) for S = 0 is the equation

$$\frac{\partial}{\partial t}\frac{\overline{T'^2}}{2} = -H\overline{T'^2} - \mu\overline{\left(\frac{\partial T'}{\partial x}\right)^2}. \tag{16}$$

If the second term on the right side of (16) is much smaller than the first, then the behavior of our system will be close to the behavior of the system with the effective dissipative function $H_1\overline{T'^2}$, where $H - \mu(2\pi/\lambda)^2 < H_1 < H + \mu(2\pi/\lambda)^2$. The formation of a discontinuity in such a system is possible for the condition $A(2\pi/\lambda)\sin(2\pi/\lambda)x < -H_1\overline{T}/U$. The formation time of a zero-magnitude discontinuity is determined in this case from the formula

$$t = -\frac{1}{H_1}\ln\left(1 - \frac{H_1\overline{T}}{A2\pi/\lambda}\right).$$

Substituting parameter values characteristic of the real atmosphere, we obtain the result that discontinuity formation is possible only for $\lambda < A2\pi U/H_1\overline{T} = 2 \times 10^6$ m. The resulting scale is an order of magnitude smaller than the scales at which the approximation being considered operates. Consequently, the slow mode of the ultralong waves, propagating in the earth's atmosphere, will be described adequately by the linearized Eq. (15).

If on the other hand, however, it is now assumed that the first term on the right side of (16) is much smaller than the second, then our system is described by the Burgers equation. In the case of severe dissipation (when $\lambda AU/4\pi\overline{T}\mu \ll 1$; this quantity is of the order of 0.05 for the earth's atmosphere) the system behavior is well described by the formula

$$T'=A\cos\frac{2\pi}{\lambda}(x-Ut)\exp\left\{-\mu\left(\frac{2\pi}{\lambda}\right)^2 t\right\},$$

giving eastward wave propagation with velocity U and simultaneous decay with a decrement $\mu(2\pi/\lambda)^2 = (23\ \text{days})^{-1}$ (for $\mu = 3 \times 10^6\ \text{m}^2/\text{sec}$ and $\lambda = 1.5 \times 10^7$ m).

6. Let us examine the case $S \neq 0$, $\mu = H = 0$ by solving the resulting equation by the method of characteristics. Ignoring the term ST', which describes the decay (the role of the latter was investigated in the previous paragraph), we write (15) in the form:

$$\frac{dT'}{dt}=-S\tau\sin kx,\quad \frac{dx}{dt}=U+\eta T', \tag{17}$$

where $\eta = U/\bar{T}$, $k = 4\pi/L_0$.

After eliminating T' from (17), we obtain for x the nonlinear equation of pendulum oscillations $d^2x/dt^2 + \eta S\tau \sin kx = 0$, which is solved in elliptic functions. Then, substituting this solution into the first of Eqs. (17), we obtain the problem solution in the form:

$$T'=-\frac{S\tau b}{\sqrt{a}}dn[b(\sqrt{a}t+f(\xi)),b^{-1}]+\frac{S\tau b}{\sqrt{a}}dn[bf(\xi),b^{-1}]+T'(\xi), \tag{18}$$

$$x=\frac{2}{k}\arcsin\{\text{sn}[b(\sqrt{a}t+f(\xi)),b^{-1}]\}. \tag{19}$$

Here

$$a=k\eta S\tau,\quad b^2=\sin^2\frac{k\xi}{2}+\frac{\nu^2}{4a},\quad \nu=k(U+\eta T'(\xi)),$$

$$f(\xi)=\frac{1}{b}\int_0^{k\xi/2}[1-b^{-2}\sin^2\varphi]^{-1/2}d\varphi,$$

$$T'(\xi)=T'(x,0).$$

The question of the possibility of discontinuity formation is reduced to the question of the existence of the envelope of the characteristic curves (18) and (19) [5, chap. 2, § 1]. This envelope must satisfy both Eq. (19) itself as well as the equation

$$\frac{db}{d\xi}\sqrt{a}t-\int_0^{k\xi/2}\sin^2\varphi[1-b^{-2}\sin^2\varphi]^{-1/2}d\varphi\frac{db}{d\xi}\frac{1}{b^3}+\frac{k/2}{\sqrt{1-b^{-2}\sin^2 k\xi/2}}=0,$$

obtained by differentiating (19) with respect to ξ. For the initial temperature profile, chosen in the form $A\sin(k\xi+\varphi_0)$ (here φ_0 is the phase shift of the initial temperature profile relative to the nonzonal drive) with amplitude A = 5 K and for a coefficient value $S\tau = 0.8 \times 10^{-5}$ K/sec, a discontinuity begins with zero magnitude at the instant of time

$$t^*\approx[kU\sqrt{b_1^2/4+b_2^2-b_1b_2\sin\varphi_0}]^{-1},$$

where $b_1 = 2S\tau/\bar{T}kU = 1.1 \times 10^{-2}$, $b_2 = A/\bar{T} = 2 \times 10^{-2}$. It is seen that the discontinuity can start earlier or later than for the adiabatic case (where $t^* = 1/kUb_2$), depending on the values of the phase shift φ_0. The minimum t* value occurs at $\varphi_0 = -\pi/2$ and is equal to ~73 days.

7. Summing up, we can conclude that the nonlinear effects, accompanying the propagation of slow ultralong temperature waves in the earth's atmosphere, are extremely weak and that these waves (subjected, as we have seen, to a strong dissipation influence) are described well within the framework of linear theory.

A different picture is observed for the atmospheres of giant planets, where large-scale processes are dynamically similar, in terms of the values of the dimensionless criteria (including the value of $\sim 10^{-2}$ for the Rossby number), to the planetary motions in the earth's atmosphere and can be described by a simplified system of equations, similar to (1). However, the nonadiabatic factors (including dissipation) are at least an order of magnitude weaker than on earth (thus, the characteristic temperature relaxation time on Jupiter amounts to hundreds of days [8]), and the nonlinear terms in the equations can manifest themselves in full force.

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REFERENCES

1. Pisnichenko, I.A. Dynamics of ultralong waves in a two-dimensional baroclinic atmospheric model. Bull. (Izv.), Acad. Sci. USSR, Atmospheric and Oceanic Physics, *16*, No. 9, 1980.

2. Kibel', I.A. Application of the equations of mechanics of a baroclinic fluid to meteorology. Izv. Akad. Nauk SSSR, Ser. Geogr. i Geofiz., *4*, No. 4, 627, 1940.

3. Obukhov, A.M. The question of the geostrophic wind. Izv. Akad. Nauk SSSR, Ser. Geogr. i Geofiz., *13*, No. 4, 281, 1949.
4. Alishayev, D.M. On the dynamics of a two-dimensional baroclinic atmosphere. Bull. (Izv.), Acad. Sci. USSR, Atmospheric and Oceanic Physics, *16*, No. 2, 1980.
5. Whisem, J. Linear and Nonlinear Waves. Mir Press, Moscow, 1977, 613 pages.
6. Landau, L.D. and Ye. M. Lifshits. Mechanics of Continuous Media. 2nd ed. Gostekhizdat Press, Moscow, 1954, 795 pages.
7. Marchuk, G.I., G.P. Kurbatkin, and V.I. Tsvetkov. Numerical model of the global atmospheric circulation. In: Numerical Methods of Solving Problems of Weather Forecasting and the General Atmospheric Circulation. Computer Center, Siberian Division, Acad. Sci. USSR, Novosibirsk, 1970, pp. 98-142.
8. Golitsyn, G.S. Introduction to the Dynamics of Planetary Atmospheres. Gidrometeoizdat Press, Leningrad, 1973, 103 pages.